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# Eigenmatrices and operators commuting with finite-rank operators

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**Abstract**

We prove, using eigenmatrices, that if an operator commutes with an operator of finite rank, then it commutes with an operator of rank one.


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## 1. Introduction

The spectrum of a linear operator is one of the most useful objects in functional analysis. Among points in the spectrum of a linear operator, eigenvalues are particularly interesting since they allow us to view the operator, restricted to the associated subspace of dimension one, simply as multiplication by a complex number. Also, existence of eigenvalues is equivalent to the existence of one-dimensional invariant subspaces.

A natural generalization of the concept of eigenvalue is to allow a subspace of larger dimension (say $n$) and view the operator acting on this subspace as an $n \times n$ matrix (called an eigenmatrix). There are several ways to do this, and the interested reader may wish to see [1,3,6–9] for further information ([6] is a good place to start). As we see below, existence of $n \times n$ eigenmatrices is equivalent to existence of $n$-dimensional invariant subspaces.

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The purpose of this note is twofold. First, we wish to introduce a slightly different notion of eigenmatrix than the one in the references above. Our definition has several disadvantages, mainly because it allows too many matrices to be eigenmatrices (in fact, if it is not empty, it will typically be an unbounded set). However, it does have a few nice properties that we feel are worthy of further investigation. For example, in the second part of this note, we will use this concept to characterize when an operator commutes with an operator of finite rank (Corollary 3.4). We use this characterization to show our main result: if an operator commutes with a finite-rank operator it must also commute with an operator of rank one (Corollary 3.6). We also show as a corollary that Toeplitz operators (on the Hardy–Hilbert space) do not commute with operators of finite rank (Corollary 3.8).

## 2. Eigenmatrices

We will introduce notation as needed. Let $\mathscr{H}$ be a complex (finite or infinite-dimensional) Hilbert space and let $A$ be an operator (a bounded linear transformation) on $\mathscr{H}$.

Also, let $\Lambda = (\lambda_{i,j})_{i,j=1}^{n}$ be an $n \times n$ complex matrix. As such, it acts on $\mathbb{C}^n$. We will also think of $\Lambda$ as acting on $\mathscr{H}^n = \mathscr{H} \oplus \mathscr{H} \oplus \cdots \oplus \mathscr{H}$ (the direct sum of $\mathscr{H}$ with itself $n$ times) as if it was the matrix $(\lambda_{i,j} I)_{i,j=1}^{n}$, with $I$ the identity operator on $\mathscr{H}$.

**Definition 2.1.** Let $A$ be an operator on $\mathscr{H}$ and $\Lambda$ be an $n \times n$ complex matrix. We say that $\Lambda$ is an *eigenmatrix* for $A$ if there exists $\Phi \in \mathscr{H}^n$ such that

$$(A \oplus A \oplus \cdots \oplus A)\Phi = \Lambda\Phi,$$

and $\Phi := (f_1, f_2, \ldots, f_n)$ has the property that $\{f_1, f_2, f_3, \ldots, f_n\}$ is a linearly independent set. We call $\Phi$ a *corresponding eigenvector* for $\Lambda$.

We denote the set of $n \times n$ eigenmatrices of $A$ by $\sigma_p^n(A)$.

Observe that the $1 \times 1$ matrix $\Lambda = (\lambda)$ is an eigenmatrix of $A$ if and only if $\lambda$ is an eigenvalue of $A$. Hence $\sigma_p^1(A) = \sigma_p(A)$.

Notice that the equation

$$(A \oplus A \oplus \cdots \oplus A)(f_1, f_2, \ldots, f_n) = \Lambda(f_1, f_2, \ldots, f_n)$$

is equivalent to the equations

$$Af_i = \sum_{j=1}^{n} \lambda_{i,j} f_j \quad \text{for } i = 1, 2, \ldots, n.$$

Therefore, existence of $n \times n$ eigenmatrices is equivalent to the existence of invariant subspaces of dimension $n$. It is clear that not all operators will have eigenmatrices.

Before presenting some examples and basic properties, we consider an alternative way of looking at the above definition.

Define a bounded linear operator $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ by assigning to each vector in the canonical basis of $\mathbb{C}^n$ a vector in $\mathscr{H}$. That is, if $\{e_1, e_2, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$ and $\{f_1, f_2, \ldots, f_n\}$ are $n$ arbitrary vectors in $\mathscr{H}$, we can define $\Theta$ as $\Theta e_i = f_i$ for each $i = 1, 2, \ldots, n$ and extend linearly. Clearly, $\Theta$ is well-defined and bounded.

Observe that we do not require the components of $\Phi$ in the following proposition to be a linearly independent set. As usual, $\Lambda^{\mathrm{T}}$ denotes the transpose of the matrix $\Lambda$.

**Proposition 2.2.** *Let $A$ be a bounded operator on $\mathscr{H}$ and let $\Lambda$ be an $n \times n$ complex matrix. Then there exists $\Phi \in \mathscr{H}^n$ such that*

$$(A \oplus A \oplus \cdots \oplus A)\Phi = \Lambda\Phi$$

*if and only if there exists $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda^{\mathrm{T}}$.*

**Proof.** For each $i$ and $j$, let $\lambda_{i,j}$ be the $(i, j)$ entry of $\Lambda$. Suppose that there is $\Phi = (f_1, f_2, \ldots, f_n) \in \mathscr{H}^n$ such that

$$(A \oplus A \oplus \cdots \oplus A)\Phi = \Lambda\Phi.$$

Then we have

$$Af_i = \sum_{j=1}^{n} \lambda_{i,j} f_j \quad \text{for } i = 1, 2, \ldots, n.$$

Define $\Theta$ by setting $\Theta e_i = f_i$, for $i = 1, 2, \ldots, n$, where $e_i$ is the $i$th canonical vector in $\mathbb{C}^n$. As observed above, $\Theta$ is a bounded operator from $\mathbb{C}^n$ into $\mathscr{H}$.

Notice that

$$A\Theta e_i = Af_i = \sum_{j=1}^{n} \lambda_{i,j} f_j.$$

Also,

$$\Theta\Lambda^{\mathrm{T}} e_i = \Theta\left(\sum_{j=1}^{n} \lambda_{i,j} e_j\right) = \sum_{j=1}^{n} \lambda_{i,j} \Theta e_j = \sum_{j=1}^{n} \lambda_{i,j} f_j.$$

Thus $A\Theta e_i = \Theta\Lambda^{\mathrm{T}} e_i$ for each $i$ and, by linearity, $A\Theta = \Theta\Lambda^{\mathrm{T}}$.

Conversely, assume that there is a bounded operator $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda^{\mathrm{T}}$. For $i = 1, 2, \ldots, n$, define $f_i \in \mathscr{H}$ by $f_i = \Theta e_i$. Then

$$Af_i = A\Theta e_i = \Theta\Lambda^{\mathrm{T}} e_i = \Theta\left(\sum_{j=1}^{n} \lambda_{i,j} e_j\right) = \sum_{j=1}^{n} \lambda_{i,j} f_j.$$

Therefore, if we set $\Phi = (f_1, f_2, \ldots, f_n)$, we have

$$(A \oplus A \oplus \cdots \oplus A)\Phi = \Lambda\Phi,$$

as desired. □

Adding the requirement of linear independence produces the following.

**Corollary 2.3.** *Let $A$ be an operator on $\mathscr{H}$. Then an $n \times n$ matrix $\Lambda$ is an eigenmatrix of $A$ if and only if there exists an injective operator $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda^{\mathrm{T}}$.*

**Proof.** It suffices to observe that in the proposition above, $\Theta$ is one-to-one if and only if $\{f_1, f_2, \ldots, f_n\}$ is linearly independent, where $f_i = \Theta e_i$. □

Note that existence of the injective operator $\Theta$ in the statement of Corollary 2.3 forces $\dim \mathscr{H} \geqslant n$. We can rephrase the corollary by saying that $\Lambda$ is an eigenmatrix for $A$ if and only if there exists $\Theta$ injective that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{A} & \mathcal{H} \\ \Theta\big\uparrow & & \big\uparrow\Theta \\ \mathbb{C}^n & \xrightarrow{\Lambda^T} & \mathbb{C}^n \end{array}$$

Thus we may view $A$, restricted to an $n$-dimensional space, as acting as if it were the matrix $\Lambda^{\mathrm{T}}$. This "embedding" of $\mathbb{C}^n$ in $\mathscr{H}$ is not necessarily isometric. (Compare this with the definition of eigenmatrix given by Davis in [3]: $\Lambda$ is an eigenmatrix if it satisfies $A\Theta = \Theta\Lambda$ for $\Theta$ an *isometry*. Corollary 2.3 is weaker, thus allowing, in principle, more matrices.) The appearance of the transposition of $\Lambda$ in Corollary 2.3 is not important: a matrix is always similar to its transpose and, as we see below, if a matrix $\Lambda$ is an eigenmatrix of $A$, then all matrices similar to $\Lambda$ are also eigenmatrices of $A$.

**Proposition 2.4.** *Let $A$ be a bounded operator on $\mathscr{H}$ and let $\Gamma$ be an invertible $n \times n$ matrix. Then $\Lambda \in \sigma_p^n(A)$ if and only if $\Gamma^{-1}\Lambda\Gamma \in \sigma_p^n(A)$.*

**Proof.** Clearly it suffices to prove one implication. So assume $\Lambda \in \sigma_p^n(A)$. Then there exists an injective linear operator $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda^{\mathrm{T}}$. We can then define a linear operator $\Omega : \mathbb{C}^n \longrightarrow \mathscr{H}$ by $\Omega = \Theta(\Gamma^{-1})^{\mathrm{T}}$. Since $\Theta$ is injective and $(\Gamma^{-1})^{\mathrm{T}}$ is invertible, it follows that $\Omega$ is injective as well. Also,

$$\begin{aligned} A\Omega &= A\Theta(\Gamma^{-1})^{\mathrm{T}} \\ &= \Theta\Lambda^{\mathrm{T}}(\Gamma^{-1})^{\mathrm{T}} \\ &= \Theta(\Gamma^{-1})^{\mathrm{T}}\Gamma^{\mathrm{T}}\Lambda^{\mathrm{T}}(\Gamma^{-1})^{\mathrm{T}} \\ &= \Omega\Gamma^{\mathrm{T}}\Lambda^{\mathrm{T}}(\Gamma^{-1})^{\mathrm{T}} \\ &= \Omega(\Gamma^{-1}\Lambda\Gamma)^{\mathrm{T}}. \end{aligned}$$

Thus $A\Omega = \Omega(\Gamma^{-1}\Lambda\Gamma)^{\mathrm{T}}$ for the injective operator $\Omega$ and hence $\Gamma^{-1}\Lambda\Gamma \in \sigma_p^n(A)$. $\square$

One consequence of the above proposition is that if $\Lambda$ is in $\sigma_p^n(A)$ so is its Jordan canonical form.

Before we describe some examples, we note that a one-sided (polynomial) spectral mapping theorem holds: if $q$ is a polynomial, then it is easily checked that

$$q(\sigma_p^n(A)) \subseteq \sigma_p^n(q(A)).$$

The reverse inclusion does not hold as the following example, kindly provided to us by the referee, shows.

**Example 2.5.** Let $q(z) = z^2$ and let $A : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ be the operator given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $q(\sigma_p^2(A)) \not\supseteq \sigma_p^2(q(A))$.

**Proof.** Let $\Lambda$ be the $2 \times 2$ matrix

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A straightforward and simple calculation shows that $\Lambda \in \sigma_p^2(q(A))$. Nevertheless, since $\Lambda \neq \Omega^2$ for every $2 \times 2$ matrix $\Omega$, it follows that $\Lambda \notin q(\sigma_p^2(A))$. $\square$

Observe that if $A$ is an algebraic operator and $q$ is an annihilating polynomial for $A$, then, since $\sigma_p^n(q(A))$ equals the zero matrix, $q(\sigma_p^n(A))$ must be the zero matrix or the empty set. Thus only those matrices annihilated by $q$ may be eigenmatrices of $A$.

**Example 2.6.** Suppose $P$ is a non-trivial projection on $\mathscr{H}$. Let $k = \dim \ker P$ and $r = \dim \operatorname{ran} P$. The set of eigenmatrices of $P$ consists of the matrices similar to diagonal matrices with 0's and 1's in the diagonal whose number of 0's is one of $\{0, 1, 2, \ldots, k\}$ (or any non-negative integer, if $k = \infty$) and whose number of 1's is one of $\{0, 1, 2, \ldots, r\}$ (or any non-negative integer, if $r = \infty$).

**Proof.** First, observe that $\sigma_p(P) = \{0, 1\}$. Since $P$ is algebraic with minimal polynomial $q(x) = x^2 - x$, all elements of $\sigma_p^n(P)$ must be annihilated by $q$. A Jordan block of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}$$

is annihilated by such $q$ if and only if it is of size 1 and $\lambda$ is 0 or 1. Thus the only possible eigenmatrices of $P$ are those similar to a diagonal matrix with 0's and 1's in the diagonal. By taking appropriate linearly independent vectors in $\ker P$ and in $\operatorname{ran} P$ and forming a corresponding eigenvector, we see that all the desired matrices are eigenmatrices. $\square$

Thus, there does not seem to be much variety in the possible eigenmatrices of projections. Nevertheless, observe that, for each $n \geqslant 2$, the set $\sigma_p^n$ is not bounded.

The next example is the backward shift operator $B$ on the Hardy–Hilbert space of analytic functions $\mathbf{H}^2$ (for basic information on $B$ and $\mathbf{H}^2$ see, for example, [4,5]). If $f \in \mathbf{H}^2$, $B$ is defined by

$$(Bf)(z) = \frac{f(z) - f(0)}{z}.$$

It is well known that $\sigma_p(B)$ is the open unit disk $\mathbb{D}$.

**Example 2.7.** The eigenmatrices of $B$ are the matrices whose Jordan canonical forms have Jordan blocks with numbers of modulus less than 1 on the diagonals and contain at most one Jordan block for each such number.

**Proof.** For $\lambda \in \mathbb{D}$, we first prove that the matrix (of arbitrary size, say $n$)

$$\Lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}$$

is an eigenmatrix for $B$.

For a non-negative integer $s$, let

$$k_\lambda^s(z) = \frac{z^s}{(1-\lambda z)^{s+1}}.$$

It is easily checked that $k_\lambda^s$ is in $\mathbf{H}^2$, that $(B-\lambda)k_\lambda^{s+1} = k_\lambda^s$ for each non-negative integer $s$, and $(B-\lambda)k_\lambda^0 = 0$. Putting all of these facts together, and setting $\Phi = (k_\lambda^0, k_\lambda^1, k_\lambda^2, \dots, k_\lambda^{n-1})$, we get

$$(B \oplus B \oplus \cdots \oplus B)\Phi = \Lambda\Phi.$$

Since $\{k_\lambda^0, k_\lambda^1, k_\lambda^2, \dots, k_\lambda^{n-1}\}$ is a linearly independent set, it follows that $\Lambda \in \sigma_p^n(B)$.

It is clear that we can repeat the procedure above with the direct sum of any finite number of blocks like the one above, as long as each block has a different $\lambda$ (two blocks with the same $\lambda$, even if they are of different size, will contain vectors which are not linearly independent).

Observe that no matrices (in Jordan form) with diagonal entries of modulus larger than or equal to 1 can be eigenmatrices, since that would imply existence of eigenvalues of the same modulus. □

In contrast to the example of the projection given earlier, the eigenmatrices of $B$ exhibit a lot of diversity. Also, note that the set $\sigma_p^n$ is not closed for any $n$. Also, it is noteworthy to observe that $B^2$ has more eigenmatrices than $B$, even though they have the same set of eigenvalues.

In the above two examples, the eigenvalues of the eigenmatrices are eigenvalues of the operator. This is no coincidence. In fact, more is true.

**Proposition 2.8.** *Let $A$ be an operator on $\mathscr{H}$ and let $\Lambda \in \sigma_p^n(A)$. Then, for every natural number $k \leqslant n$,*

$$\sigma_p^k(\Lambda) \subset \sigma_p^k(A).$$

*In particular, the eigenvalues of $\Lambda$ are also eigenvalues of $A$.*

**Proof.** Since $\Lambda \in \sigma_p^n(A)$, the similar matrix $\Lambda^{\mathrm{T}}$ is also in $\sigma_p^n(A)$. Thus there exists an injective operator $\Theta : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda$.

Let $\Gamma \in \sigma_p^k(\Lambda)$. Then, by Corollary 2.3, there exists an injective operator $\Omega : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ such that $\Lambda\Omega = \Omega\Gamma^{\mathrm{T}}$.

Thus we have $A\Theta\Omega = \Theta\Lambda\Omega = \Theta\Omega\Gamma^{\mathrm{T}}$, and, since $\Theta\Omega : \mathbb{C}^k \longrightarrow \mathscr{H}$ is injective, we conclude that $\Gamma \in \sigma_p^k(A)$. □

Observe that the result above continues to be true if we use Davis's definition.

## 3. Commuting with a finite-rank operator

Recall that for $f$ and $g$ in $\mathscr{H}$, the operator $f \otimes g$ is defined by

$$(f \otimes g)h = (h, g)f \quad \text{for all } h \in \mathscr{H}.$$

It is easily shown that, if $A$ and $B$ are bounded operators on $\mathscr{H}$, then $A(f \otimes g)B = (Af) \otimes (B^*g)$. Also, $f \otimes g = 0$ if and only if $f = 0$ or $g = 0$. For each natural number $n$, an operator of rank $n$ is always of the form $\sum_{i=1}^{n} f_i \otimes g_i$, for vectors $f_i$ and $g_i \in \mathscr{H}$ such that $\{f_1, f_2, \ldots, f_n\}$ and $\{g_1, g_2, \ldots, g_n\}$ are two linearly independent sets.

The following proposition is undoubtedly well known. We include a simple proof here for completeness.

**Proposition 3.1.** *Let $A$ be a bounded operator on $\mathscr{H}$ and let $f$ and $g$ be nonzero vectors in $\mathscr{H}$. Then $A$ commutes with $f \otimes g$ if and only if $Af = \lambda f$ and $A^*g = \overline{\lambda} g$ for some $\lambda \in \mathbb{C}$.*

**Proof.** Assume that $A(f \otimes g) = (f \otimes g)A$. Then it follows that $(Af) \otimes g = f \otimes (A^*g)$. Since $g \neq 0$, this occurs only if there exists $\lambda \in \mathbb{C}$ such that $Af = \lambda f$. Substituting this into the above equation results in $(\lambda f) \otimes g = f \otimes A^*g$ and hence, since $f \neq 0$, we obtain $A^*g = \bar{\lambda} g$, as desired.

Conversely, assume that $Af = \lambda f$ and $A^*g = \bar{\lambda} g$. Then,

$$A(f \otimes g) = (Af) \otimes g = (\lambda f) \otimes g = \lambda(f \otimes g).$$

Analogously,

$$(f \otimes g)A = f \otimes (A^*g) = f \otimes (\bar{\lambda} g) = \lambda(f \otimes g).$$

Thus $A$ commutes with $f \otimes g$. $\square$

If $F \subset \mathbb{C}$, we define $F^*$ as the set of the complex conjugates of the elements of $F$. The following corollary is immediate.

**Corollary 3.2.** *A bounded operator $A$ commutes with a rank-one operator if and only if $\sigma_p(A) \cap (\sigma_p(A^*))^* \neq \varnothing$.*

The following extends the above characterization to operators commuting with operators of finite rank. As usual, $\Lambda^*$ denotes the adjoint (i.e., conjugate-transpose) of $\Lambda$.

**Proposition 3.3.** *Let $\{f_1, f_2, \ldots, f_n\}$ and $\{g_1, g_2, \ldots, g_n\}$ be two linearly independent sets of vectors in $\mathscr{H}$ and let $A$ be a bounded operator on $\mathscr{H}$. Then $A$ commutes with $\sum_{i=1}^{n} f_i \otimes g_i$ if and only if there exists an $n \times n$ matrix $\Lambda$ such that*

$$(A \oplus A \oplus \cdots \oplus A)(f_1, f_2, \ldots, f_n) = \Lambda(f_1, f_2, \ldots, f_n),$$

*and*

$$(A^* \oplus A^* \oplus \cdots \oplus A^*)(g_1, g_2, \ldots, g_n) = \Lambda^*(g_1, g_2, \ldots, g_n).$$

**Proof.** Suppose first that $A$ commutes with $\sum_{i=1}^{n} f_i \otimes g_i$. Then we have

$$\sum_{i=1}^{n} (Af_i) \otimes g_i = \sum_{i=1}^{n} f_i \otimes (A^*g_i). \tag{1}$$

Since $\{g_1, g_2, \ldots, g_n\}$ is linearly independent, there exist complex numbers $\lambda_{i,j}$ such that

$$Af_i = \sum_{j=1}^{n} \lambda_{i,j} f_j, \quad \text{for } i = 1, 2, \ldots, n. \tag{2}$$

Substituting Eq. (2) into Eq. (1) gives

$$\sum_{i=1}^{n} \left( \sum_{j=1}^{n} \lambda_{i,j} f_j \right) \otimes g_i = \sum_{i=1}^{n} f_i \otimes (A^* g_i).$$

Hence

$$\sum_{j=1}^{n} f_j \otimes \left( \sum_{i=1}^{n} \overline{\lambda_{i,j}} g_i \right) = \sum_{i=1}^{n} f_i \otimes (A^* g_i).$$

Renaming indices, we obtain

$$\sum_{i=1}^{n} f_i \otimes \left( \sum_{j=1}^{n} \overline{\lambda_{j,i}} g_j \right) = \sum_{i=1}^{n} f_i \otimes (A^* g_i).$$

Since $\{f_1, f_2, \ldots, f_n\}$ is linearly independent it follows that

$$A^* g_i = \sum_{j=1}^{n} \overline{\lambda_{j,i}} g_j \quad \text{for } i = 1, 2, \ldots, n. \tag{3}$$

Observe that Eqs. (2) and (3) are equivalent to

$$(A \oplus A \oplus \cdots \oplus A)(f_1, f_2, \ldots, f_n) = \Lambda(f_1, f_2, \ldots, f_n),$$

and

$$(A^* \oplus A^* \oplus \cdots \oplus A^*)(g_1, g_2, \ldots, g_n) = \Lambda^*(g_1, g_2, \ldots, g_n),$$

with $\Lambda = (\lambda_{i,j})_{i,j=1}^{n}$. Thus we have proven necessity.

To prove sufficiency is a matter of substitution. From

$$(A \oplus A \oplus \cdots \oplus A)(f_1, f_2, \ldots, f_n) = \Lambda(f_1, f_2, \ldots, f_n),$$

and

$$(A^* \oplus A \oplus \cdots \oplus A^*)(g_1, g_2, \ldots, g_n) = \Lambda^*(g_1, g_2, \ldots, g_n),$$

we obtain

$$Af_i = \sum_{j=1}^{n} \lambda_{i,j} f_j, \quad \text{and} \quad A^* g_i = \sum_{j=1}^{n} \overline{\lambda_{j,i}} g_j \quad \text{for } i = 1, 2, \ldots, n.$$

Hence

$$A \left( \sum_{i=1}^{n} f_i \otimes g_i \right) = \sum_{i=1}^{n} (Af_i) \otimes g_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} \lambda_{i,j} f_j \right) \otimes g_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} f_j \otimes g_i.$$

Analogously,

$$\left( \sum_{i=1}^{n} f_i \otimes g_i \right) A = \sum_{i=1}^{n} f_i \otimes (A^* g_i) = \sum_{i=1}^{n} f_i \otimes \left( \sum_{j=1}^{n} \overline{\lambda_{j,i}} g_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{j,i} f_i \otimes g_j.$$

Thus

$$A\left(\sum_{i=1}^{n} f_i \otimes g_i\right) = \left(\sum_{i=1}^{n} f_i \otimes g_i\right) A,$$

which finishes the proof. □

We can now generalize Corollary 3.2. For $M$ a set of $n \times n$ matrices, we denote by $M^*$ the set of all adjoints (i.e., conjugate-transposes) of matrices in $M$.

**Corollary 3.4.** *A bounded operator $A$ commutes with a rank $n$ operator if and only if $\sigma_p^n(A) \cap (\sigma_p^n(A^*))^* \neq \varnothing$.*

**Proof.** This follows immediately from the previous proposition by observing that $\sum_{i=1}^{n} f_i \otimes g_i$ is an operator of rank $n$ if and only if $\{f_1, f_2, \ldots, f_n\}$ and $\{g_1, g_2, \ldots, g_n\}$ are two linearly independent sets. □

The above corollary is false if one uses Davis's definition of eigenmatrices. Indeed, if one requires the components of the eigenvector $\Phi$ (see Definition 2.1) to be an orthonormal set instead of just being linearly independent one can construct examples where Corollary 3.4 does not hold.

Observe that Corollary 3.2 implies that operators on a finite-dimensional Hilbert space always commute with operators of rank 1, since in that case $\sigma_p(A) = (\sigma_p(A^*))^*$ and $\sigma_p(A)$ is always non-empty. If we are dealing with infinite-dimensional $\mathscr{H}$, then one is tempted to use a similar argument to show that if $A$ commutes with an operator of finite rank, then it commutes with an operator of rank 1. Indeed, if $AF = FA$ for a finite-rank operator $F$, then $\mathscr{M} = \operatorname{clos}\operatorname{ran} F$ is a (finite-dimensional) invariant subspace of $A$. Hence $A$ has an eigenvalue, say $\lambda$. Does it follow that $\bar{\lambda}$ is an eigenvalue of $A^*$? Surprisingly, the answer is yes for at least one eigenvalue of $A$. Furthermore, the following is true.

**Theorem 3.5.** *Let $A$ be a bounded operator on $\mathscr{H}$ that commutes with an operator of (finite) rank $n$ and let $\Lambda$ be in $\sigma_p^n(A) \cap (\sigma_p^n(A^*))^*$. If $s$ is the number of blocks in the Jordan canonical form of $\Lambda$, then, for each $t = 1, 2, \ldots, s$, the intersection $\sigma_p^t(A) \cap (\sigma_p^t(A^*))^*$ is non-empty. Equivalently, for each $t = 1, 2, \ldots, s$, there is an operator of rank $t$ that commutes with $A$.*

**Proof.** The existence of $\Lambda$ was established in Corollary 3.4. As we have observed before, we may assume that $\Lambda$ is in its Jordan canonical form.

Note that, for each Jordan block $i$, where $i = 1, 2, \ldots, s$, there is a row with all non-diagonal entries equal to zero. Let $\lambda_i \in \mathbb{C}$ be the diagonal entry corresponding to that row. Then, if $h_i \in \mathscr{H}$ is the entry of the eigenvector (associated with $\Lambda$) corresponding to such a row, it follows that $Ah_i = \lambda_i h_i$. By the definition of eigenmatrix, the set $\{h_1, h_2, \ldots, h_s\}$ is linearly independent.

Since $\Lambda^* \in \sigma_p^n(A^*)$, we know that any matrix similar to $\Lambda^*$ is an eigenmatrix of $A^*$. Thus the Jordan canonical form of $\Lambda^*$ is also an eigenmatrix of $A^*$. Observe that, except for taking conjugates of the diagonal elements, the Jordan canonical form of $\Lambda^*$ is the same as that of $\Lambda$ (this is immediate from the fact that every matrix is similar to its transpose).

Renaming $\Lambda^*$ if necessary, it follows that there is $\Psi \in \mathscr{H}^n$ such that

$$(A^* \oplus A^* \oplus \cdots \oplus A^*)\Psi = \Lambda^* \Psi,$$

with $\Lambda^*$ in its Jordan canonical form.

Proceeding as at the beginning of this proof, we observe that, for each Jordan block $i$, there exists a nonzero $k_i \in \mathscr{H}$ such that $A^*k_i = \overline{\lambda_i}k_i$. By the definition of eigenmatrix, the set $\{k_1, k_2, \ldots, k_s\}$ is linearly independent.

Clearly, each rank-one operator $h_i \otimes k_i$ commutes with $A$, and hence the sum of any $t$ of them ($t = 1, 2, \ldots, s$) commutes with $A$. But each such sum is of rank $t$ (since $\{h_1, h_2, \ldots, h_s\}$ and $\{k_1, k_2, \ldots, k_s\}$ are linearly independent). Hence the theorem is proved. □

The following quite unexpected corollary follows.

**Corollary 3.6.** *Every operator that commutes with a nonzero finite-rank operator commutes with an operator of rank one.*

Also, the case $t = 1$ of Theorem 3.5 contains the following result.

**Corollary 3.7.** *If $A$ commutes with a nonzero finite-rank operator, then there exists $\lambda \in \sigma_p(A)$ such that $\bar{\lambda} \in \sigma_p(A^*)$.*

Recall that a nonzero Toeplitz operator $T$ on $\mathbf{H}^2$ has the property (by Coburn's alternative [4, p. 185]) that if $\lambda$ is an eigenvalue of $T$, then $\bar{\lambda}$ cannot be an eigenvalue of $T^*$. Thus Toeplitz operators cannot commute with operators of rank one. The above corollary immediately yields the following.

**Corollary 3.8.** *No nonzero Toeplitz operator commutes with a finite-rank operator.*

This contrasts with the fact that there are Toeplitz operators (see [2]) that commute with compact operators.

Proposition 3.3 can also be proven by using Corollary 2.3. Indeed, assume that $\Lambda$ is an eigenmatrix for $A$ and $\Lambda^*$ is an eigenmatrix for $A^*$. Then there exist injective operators $\Theta$ and $\Omega : \mathbb{C}^n \longrightarrow \mathscr{H}$ such that $A\Theta = \Theta\Lambda^{\mathrm{T}}$ and $A^*\Omega = \Omega\Lambda^{*\mathrm{T}}$. But then $A\Theta\Omega^* = \Theta\Lambda^{\mathrm{T}}\Omega^*$ and $A^*\Omega\Theta^* = \Omega\Lambda^{*\mathrm{T}}\Theta^*$. Hence $A\Theta\Omega^* = (A^*\Omega\Theta^*)^*$. Since $\Theta\Omega^* : \mathscr{H} \longrightarrow \mathscr{H}$ is an operator of rank $n$ and $A\Theta\Omega^* = \Theta\Omega^*A$, it follows that $A$ commutes with an operator of rank $n$. The converse is proven similarly.

**Note:** The results in Section 2 of this note clearly hold for operators on an arbitrary Banach space $X$. The results in Section 3 also hold, under the following caveats: one must observe that the finite-rank operators on $X$ can be naturally identified with the algebraic tensor product $X \otimes X^*$, where $X^*$ is the (Banach space) dual of $X$; one must consider the Banach-space adjoints (i.e., without complex conjugates) of the operators and matrices involved, instead of the Hilbert-space adjoints (with complex conjugates); and one must remove all of the complex-conjugate signs appearing on any (complex) number.

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